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Symmetry-adapted equilibrium matrices

R.D. Kangwai, S.D. Guest*

Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK

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Abstract

A number of authors have previously used group representation theory to block-diagonalise the stiffness matrix of a symmetric structure. This paper describes how similar techniques can be used to block-diagonalise the equilibrium matrix of a symmetric structure. This is shown to provide useful insight into the static and kinematic response of such systems. In particular, it simplifies finding and classifying states of self-stress and mechanisms, as well as reducing the computational effort required for a Force Method analysis. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Many authors have applied group representation theory to the analysis of symmetric structures. This work has usually been based on the Stiffness Method of structural analysis, and hence these methods essentially provide a way to block-diagonalise a stiffness matrix into submatrix blocks corresponding to particular symmetry properties of the structure. Typical examples of this technique include Zhong and Qui (1983), Dinkevich (1991), Healey and Treacy (1991), Zloković (1992); this large body of work has recently been reviewed and explained in Kangwai et al. (1999).

Very few authors, however, have considered the impact of group representation theory on the equilibrium relationships between external forces and internal stress resultants for a symmetric structure, or, equivalently, the compatibility relationships between external joint displacements, and internal deformations. It is precisely this impact that the current paper addresses by showing the simplification that group representation theory can bring to the linearised equilibrium or compatibility relationships for a structure. Essentially the paper shows how an equilibrium matrix can be block-diagonalised into submatrix blocks which relate external forces and internal stress resultants with particular symmetry properties. An important advantage of this method is that all the work done on the implications of the

* Corresponding author. Tel.: +44-01223-332708; fax: +44-01223-332662.
E-mail address: sdg@eng.cam.ac.uk (S.D. Guest)

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linear algebra of equilibrium matrices, e.g. Strang (1986, chap. 2), Calladine (1978), Pellegrino and Calladine (1986), Pellegrino (1993) is equally applicable to these submatrix blocks. The method also allows considerable simplifications to be made in a Force Method analysis of a symmetric structure.

A key feature of the work described in this paper is finding an internal symmetry-adapted coordinate system for a structure, suitable for expressing the stress resultants. This has many similarities to previous work where inter-atomic distances and angles between chemical bonds have been used to define an internal coordinate system for a symmetric molecule, see, for example, Wilson et al. (1955).

The work described in this paper builds on the mathematical foundations described by Bossavit (1993). Bossavit showed that group representation theory could be applied to equilibrium and compatibility relationships for symmetric structures (note however, that Bossavit's compatibility matrix might be better described as an incompatibility matrix, whose row space corresponds to the left-nullspace of the compatibility matrix used in this paper). The current paper considers many of the practical implications of this mathematical foundation. In particular, it puts it into the context of earlier work on the linear algebra of the equilibrium matrix, showing the implications for states of self stress and mechanisms in a structure, and the application to Force Method analysis.

Zloković (1989) has previously considered the effect of symmetry on a Force (or Flexibility) Method analysis of a structure, but used a quite different approach to the one in this paper. In particular, the effect of symmetry is not considered directly on equilibrium relationships; rather a statically indeterminate structure is made determinate by sets of fully symmetric 'cuts' in the structure, with equilibrium being a subsequent step to this. This approach would have difficulties when applied to, for example, the example structures in this paper, where there does not exist a fully symmetric set of 'cuts' that will make the structures determinate. Also, Zloković's approach is based on the characters of irreducible representations, rather than the representations themselves, and so is not able directly to split subspaces corresponding to more-than-one dimensional irreducible representations, although this is sometimes rectified by ad-hoc methods.

The layout of this paper is as follows. Section 2 outlines the equilibrium, compatibility and flexibility relationships required for the static analysis of structures. Section 3 shows how symmetry-adapted coordinate systems can be found for both internal and external vector spaces. Section 4 shows that by using the symmetry subspaces to provide symmetry-adapted coordinate systems, the equilibrium matrix for any symmetric structure is block-diagonalised into submatrix blocks with particular symmetry properties. Section 5 describes how block-diagonalising the equilibrium matrix facilitates the identification of states of self-stress and mechanisms present in the structure, and shows the simplification that is possible for a Force Method analysis of a structure. Section 6 provides a detailed example, where the equilibrium matrix of a tensegrity dome, originally analysed by Pellegrino (1992), is block-diagonalised to simplify the identification of states of self-stress and mechanisms in the structure. Section 7 concludes the paper.

2. Equilibrium, compatibility and flexibility matrices

Traditional structural analysis requires three principles to be satisfied; that internal forces are in equilibrium with the applied load, that any internal deformation is compatible with external displacements, and that internal forces and displacements are related by a material law.

For small perturbations around the initial configuration of a structure, these relationships can be linearised as three matrix relationships.

The system of static equilibrium equations for a general structure is given by:

$$\mathbf{Hf} = \mathbf{p} \tag{1}$$

where **H** is the equilibrium matrix, **f** is the internal force vector, **p** is the external load vector.

The system of kinematic compatibility equations is given by:

$$\mathbf{Cd} = \mathbf{e} \tag{2}$$

where **C** is the compatibility matrix, **d** is the external displacement vector, **e** is the internal deformation vector.

The stress–strain relationship is given by:

$$\mathbf{Rf} = \mathbf{e} \tag{3}$$

where **R** is the flexibility matrix.

The solution of a problem in structural analysis requires the simultaneous solution of eqns (1)–(3). Commonly, using the Stiffness Method of structural analysis, the internal forces are condensed out, and the three sets of equations are combined to form a single stiffness relationship. However, in the Force Method of structural analysis, all three equations are used explicitly.

It can easily be shown by a virtual work argument that $\mathbf{C} = \mathbf{H}^{\mathrm{T}}$ (McGuire and Gallagher, 1979). Due to this static-kinematic duality, it is possible to analyse the equilibrium equation of a structure in order to identify both the states of self-stress present in a statically indeterminate structure and also the presence of inextensional mechanisms where rigid body motion is possible for all or part of a kinematically indeterminate structure (Pellegrino and Calladine, 1986).

The examples in this paper are restricted to the static analysis of pin-jointed structures. This is to simplify the matrices involved in the calculations, although the techniques used are equally suited to the analysis of more general structures.

3. Symmetry-adapted coordinate systems

This section described how the application of group representation theory can be used to find symmetry-adapted coordinate systems for a structure. These symmetry-adapted coordinate systems provide a basis for vector symmetry subspaces, each of which have particular symmetry properties of

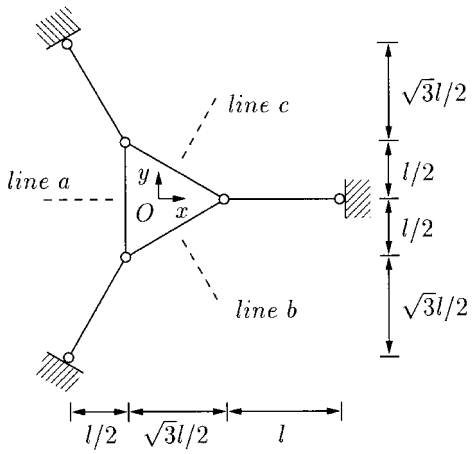

Fig. 1. Pin-jointed structure in 2-D space with $\mathbf{C}_{3v}$ symmetry. All joints are pinned.

Table 1
Irreducible representations of symmetry group $\mathbf{C}_{3v}$

$\mathbf{C}_{3v}$	E	C_3	C_3^2	σ_a	σ_b	σ_c
$\Gamma^{(A_1)}$	1	1	1	1	1	1
$\Gamma^{(A_2)}$	1	1	1	−1	−1	−1
$\Gamma^{(E)}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

the structure. This technique is well-known in structural mechanics when it is applied to an external coordinate system, suitable for representing e.g. applied loads, or joint displacements, and so is not presented in detail in this paper. Finding a symmetry-adapted coordinate system for an internal coordinate system that may be used for representing e.g. stress resultants, or bar deformation, however, is a novel technique for structural mechanics, although it is commonly applied in Chemistry in the analysis of symmetric molecules. Thus, this paper described how to find a symmetry adapted coordinate system in more detail.

The method of finding symmetry-adapted coordinate systems will be outlined with reference to the simple example structure shown in Fig. 1. This 2-D structure has $\mathbf{C}_{3v}$ symmetry, as it is unchanged by six symmetry operations: the identity, symmetry operation E; rotation by 120 or 240° about the origin O, symmetry operations C_3, C_3^2; reflection in line a, b or c, symmetry operations σ_a, σ_b or σ_c. A set of irreducible representations Γ^{μ} are given in Table 1 for $\boldsymbol{C}_{3v}$; these irreducible representations provide the key to finding symmetry adapted coordinate systems, and may be found in books of group theory tables, e.g. Altmann and Herzig (1994).

3.1. Symmetry-adapted external coordinate system

In Fig. 2 a coordinate system has been attached to the example structure that is suitable for

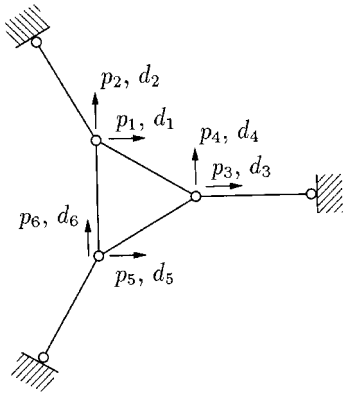

Fig. 2. A Cartesian coordinate system for the external load and displacement vector space $\mathbb{V}_{\mathrm{p}}$.

representing both load vectors **p**, and displacement vectors **d**. We use the symbol $\mathbb{V}_p$ to denote the 6-D vector space occupied by these vectors.

As has been shown by e.g. Kangwai et al. (1999) or Healey and Treacy (1991), this vector space can be split into symmetry subspaces which each correspond to a row of one of the irreducible representations of the group. For the example structure, there are four symmetry subspaces; the first two correspond to the two 1-D irreducible matrix representations, and the second two correspond to the two rows of the 2-D irreducible matrix representation:

$$\mathbb{V}_p = \left(\mathbb{V}_p^{(A_1)} \mid \mathbb{V}_p^{(A_2)} \mid \mathbb{V}_p^{(E)_1} \mid \mathbb{V}_p^{(E)_2}\right) \tag{4}$$

The symmetry adapted basis of the external symmetry subspaces are shown in Fig. 3. Each of the symmetry subspaces corresponds to a particular type of symmetry of the example structure. Any vector in the symmetry subspace $\mathbb{V}_p^{(A_1)}$ is left unchanged by any symmetry operation of the symmetry group $\mathbf{C}_{3v}$, which explains why the irreducible representation $\Gamma^{(A_1)}$ is [1] for any symmetry operation. Any vector in the symmetry subspace $\mathbb{V}_p^{(A_2)}$ is left unchanged by any rotation, but is reversed by any reflection, and so the irreducible representation $\Gamma^{(A_2)}$ is [1] for a rotation, but [−1] for a reflection. Any vector in $\mathbb{V}_p^{(E)_1}$ has only reflective symmetry in the plane a, while any vector in $\mathbb{V}_p^{(E)_2}$ has only anti-

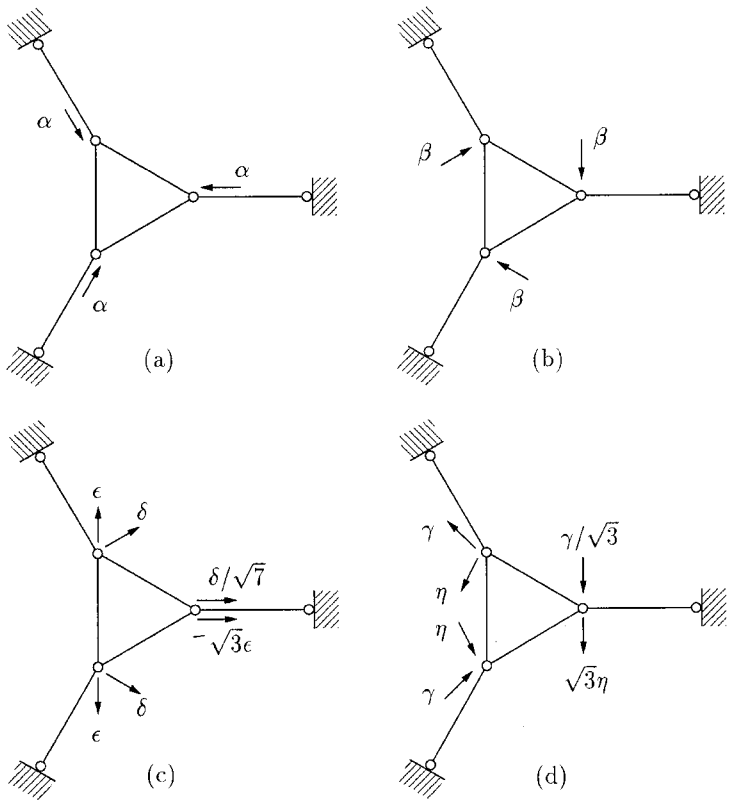

Fig. 3. External load and displacement vector symmetry subspaces: (a) $\mathbb{V}_p^{(A_1)}$; (b) $\mathbb{V}_p^{(A_2)}$; (c) $\mathbb{V}_p^{(E)_1}$; (d) $\mathbb{V}_p^{(E)_2}$. $\alpha - \gamma$ are arbitrary constants.

symmetry in the plane a. It can be seen why $\Gamma^{(E)}$ is a 2-D irreducible representation, as the effect of symmetry operations on this symmetry subspaces is more complex, with a coupling between the basis vectors.

$\mathbb{V}_{\mathrm{p}}^{(E)_1}$ and $\mathbb{V}_{\mathrm{p}}^{(E)_2}$ contain vectors which are symmetric and anti-symmetric in plane a because of the particular choice for the 2-D irreducible representation $\Gamma^{(E)}$ that we made. While the 1-D irreducible representations are unique, there is a choice of vector basis for any 2-D irreducible representation, and a different choice for $\Gamma^{(E)}$ could, for instance, have made vectors in $\mathbb{V}_{\mathrm{p}}^{(E)_1}$ and $\mathbb{V}_{\mathrm{p}}^{(E)_2}$ symmetric and anti-symmetric in a different plane.

The matrix $\mathbf{V}_{\mathrm{p}}$ gives the symmetry adapted basis in terms of the original Cartesian coordinate system

$$\mathbf{V}_p = \left[\begin{array}{c|c|cc|cc} 1/\sqrt{12} & 1/2 & \sqrt{5/12} & 0 & -1/\sqrt{20} & -1/\sqrt{5} \\ -1/2 & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} & -\sqrt{3/20} & \sqrt{4/15} \\ -1/\sqrt{3} & 0 & 1/\sqrt{15} & -\sqrt{3/5} & 0 & 0 \\ 0 & -1/\sqrt{3} & 0 & 0 & -\sqrt{3/5} & -1/\sqrt{15} \\ 1/\sqrt{12} & -1/2 & \sqrt{5/12} & 0 & 1/\sqrt{20} & 1/\sqrt{5} \\ 1/2 & 1/\sqrt{12} & -1/\sqrt{20} & -1/\sqrt{5} & -\sqrt{3/20} & \sqrt{4/15} \end{array}\right] \tag{5}$$

3.2. Symmetry-adapted internal coordinate system

It is also possible to find a symmetry adapted coordinate system that is suitable for representing quantities such as internal bar forces or extensions. As this technique is not well-known for structural mechanics, the techniques used to find this symmetry adapted coordinate system will be presented more carefully, again with particular reference to the example structure.

Fig. 4 shows a 'natural' coordinate system for both the bar-force vectors **f**, and bar-elongation vectors **e** in the example structure. This coordinate system is necessarily restricted by the configuration of the

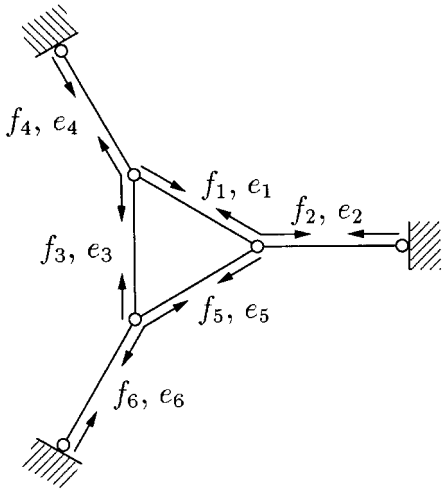

Fig. 4. A natural coordinate system for the internal bar-force and bar-elongation vector space $\mathbb{V}_{\mathrm{f}}$.

Table 2
Reducible matrix representation $\mathbf{F}$ of symmetry group $\mathbf{C}_{3v}$

$$\mathbf{F}(E)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} \qquad \mathbf{F}(\sigma_a)=\begin{bmatrix}0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}$$

$$\mathbf{F}(C_3)=\begin{bmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\end{bmatrix} \qquad \mathbf{F}(\sigma_b)=\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\end{bmatrix}$$

$$\mathbf{F}(C_3^2)=\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\end{bmatrix} \qquad \mathbf{F}(\sigma_c)=\begin{bmatrix}1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{bmatrix}$$

structure. We use the symbol $\mathbb{V}_{\mathrm{f}}$ to denote the vector space defined by all possible internal bar-forces and bar-elongations. This vector space can also be split into symmetry subspaces.

The first step in finding a symmetry-adapted coordinate system is to write a 'reducible' representation $\mathbf{F}$ for each symmetry operation, as it is represented in the original coordinate system. For the natural internal coordinate system of the example structure, this representation is given in Table 2; each of the representation matrices for the internal coordinate system of a pin-jointed structure will be a permutation matrix, as each symmetry operation will simply shuffle the numbering of the bars.

For the matrix representation $\mathbf{F}$, which operates on the vector space $\mathbb{V}_{\mathrm{f}}$ of a general structure, any standard text on group representation theory, or the papers described in the Introduction, will show that each of the symmetry subspaces $\mathbb{V}_{\mathrm{f}}^{(\mu)_i}$ are given by the column space of the projection operator matrix:

$$\mathbf{O}_{ij}^{(\mu)}=\sum_{F}\Gamma_{i,j}^{(\mu)}\mathbf{F} \tag{6}$$

where the irreducible matrix representations of the symmetry group $\mathbf{C}_{3v}$ are given in Table 1 and the summation is over the matrix representation $\mathbf{F}$ given in Table 2. For example, the symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_1)}$ is given by the column space of:

$$\mathbf{O}_{11}^{(A_1)}=\sum_{\mathrm{F}}\Gamma_{1,1}^{(A_1)}\mathbf{F}$$

$$=\left[1\times\mathbf{F}(E)+1\times\mathbf{F}(C_3)+1\times\mathbf{F}\left(C_3^2\right)+1\times\mathbf{F}(\sigma_a)+1\times\mathbf{F}(\sigma_b)+1\times\mathbf{F}(\sigma_c)\right]$$

$$= \begin{bmatrix} 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} \end{bmatrix} \tag{7}$$

A suitable vector basis $\mathbf{V}_{\mathrm{f}}^{(A_1)}$ for the symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_1)}$, can then be chosen:

$$\mathbf{V}_{\mathrm{f}}^{(A_1)} = \text{basis for column space of } \mathbf{O}_{11}^{(A_1)} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} \tag{8}$$

A similar calculation is carried out to find the vector basis for the symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_2)}$:

$$\mathbf{O}_{11}^{(A_2)} = \sum_{\mathrm{F}} \Gamma_{1,1}^{(A_2)} \mathbf{F}$$

$$= \left[1 \times \mathbf{F}(E) + 1 \times \mathbf{F}(C_3) + 1 \times \mathbf{F}\left(C_3^2\right) - 1 \times \mathbf{F}(\sigma_a) - 1 \times \mathbf{F}(\sigma_b) - 1 \times \mathbf{F}(\sigma_c) \right]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

The symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_2)}$ is given by the column space of $\mathbf{O}_{11}^{(A_2)}$ and hence is a zero space, the significance of which will be explained later.

The vector bases for the two remaining symmetry subspaces $\mathbb{V}_{\mathrm{f}}^{(E)_1}$ and $\mathbb{V}_{\mathrm{f}}^{(E)_2}$ are found in a similar way, finally giving a new symmetry-adapted basis for the bar-force and bar-elongation vector space $\mathbb{V}_{\mathrm{f}}$:

$$\mathbf{V}_{\mathrm{f}} = \left[\mathbf{V}_{\mathrm{f}}^{(A_1)} \middle| \mathbf{V}_{\mathrm{f}}^{(A_2)} \middle| \mathbf{V}_{\mathrm{f}}^{(E)_1} \middle| \mathbf{V}_{\mathrm{f}}^{(E)_2} \right] \tag{10}$$

$$\mathbf{V}_{\mathrm{f}} = \left[\begin{array}{cc||cc|cc} 1/\sqrt{3} & 0 & 0 & 1/\sqrt{6} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{3} & -\sqrt{2/3} & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 & -\sqrt{2/3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{6} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 0 & 0 & 1/\sqrt{6} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{3} & 1/\sqrt{6} & 0 & 1/\sqrt{2} & 0 \end{array}\right] \tag{11}$$

The basis vectors of the four symmetry subspaces $\mathbb{V}_{\mathrm{f}}^{(\mu)_i}$, are shown in Fig. 5. We can see the symmetry properties of each bar-force and bar-elongation vector symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(\mu)_i}$ are identical to the corresponding load and displacement vector symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(\mu)_i}$ shown in Fig. 3. In particular, we found that the second symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_2)}$ was a zero space. That is, there are no bar-force or bar-elongation vectors which have only the rotational symmetry of the example structure. In this case any bar-force or bar-elongation vector left unchanged by the rotation operations, will also be left unchanged by the reflection operations and hence belong to the first symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_1)}$.

Thus, although the corresponding symmetry subspaces $\mathbb{V}_{\mathrm{p}}^{(\mu)_i}$ and $\mathbb{V}_{\mathrm{f}}^{(\mu)_i}$ have identical symmetry properties, it is also clear that the corresponding vector bases may well have different dimensions.

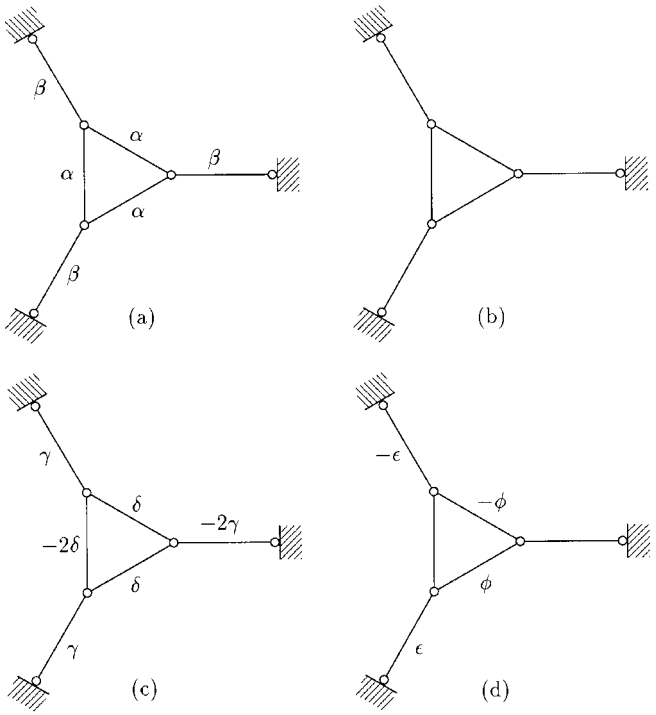

Fig. 5. Internal bar-force and bar-elongation vector symmetry subspaces: (a) $\mathbb{V}_{\mathrm{f}}^{(A_1)}$; (b) $\mathbb{V}_{\mathrm{f}}^{(A_2)}$; (c) $\mathbb{V}_{\mathrm{f}}^{(E)_1}$; (d) $\mathbb{V}_{\mathrm{f}}^{(E)_2}$. α–ϕ are arbitrary constants.

4. Block-diagonal form of the equilibrium, compatibility and flexibility matrices

Using group representation theory to exploit the full symmetry of the example structure, it has been possible to decompose the two vector spaces $\mathbb{V}_p$ and $\mathbb{V}_f$, into symmetry subspaces $\mathbb{V}_p^{(\mu)_i}$ and $\mathbb{V}_f^{(\mu)_i}$. Each of these symmetry subspaces corresponds to a different type of symmetry.

In Kangwai et al. (1999), symmetry arguments were used to show that load vectors with the symmetry properties of a particular symmetry subspace $\mathbb{V}_p^{(\mu)_i}$, would induce displacement vectors with the same symmetry properties and hence from the same symmetry subspace $\mathbb{V}_p^{(\mu)_i}$. Although the induced bar-force and bar-elongation vectors occupy a different vector space $\mathbb{V}_f$, to that of the load and displacement vectors, namely $\mathbb{V}_p$, these symmetry arguments can be extended to the equilibrium matrix.

In a stable linear system, any induced bar-force and bar-elongation vectors will have the same symmetry properties as the applied load vector. Hence, any load vector with the particular symmetry properties of symmetry subspace $\mathbb{V}_p^{(\mu)_i}$, will induce bar-force and bar-elongation vectors from the corresponding symmetry subspace $\mathbb{V}_f^{(\mu)_i}$, which has the same symmetry properties. Therefore, the equilibrium matrix $\mathbf{H}$ can be block-diagonalised into independent submatrix blocks, each operating on a pair of corresponding symmetry subspaces $\mathbb{V}_p^{(\mu)_i}$ and $\mathbb{V}_f^{(\mu)_i}$. The compatibility matrix $\mathbf{C}$ and flexibility matrix $\mathbf{R}$ can also be block-diagonalised in a similar way.

The key to the block-diagonalisation is to define the load vectors $\tilde{\mathbf{p}}$ in the symmetry-adapted vector basis $\mathbf{V}_p$, and the bar-force vectors $\tilde{\mathbf{f}}$ in the symmetry-adapted vector basis $\mathbf{V}_f$ (a $\sim$ is used for symmetry-adapted systems). These can be transformed into equivalent load and bar-force vectors $\mathbf{p}$ and $\mathbf{f}$ in the original coordinate systems, by the following transformations:

$$\mathbf{p} = \mathbf{V}_p\tilde{\mathbf{p}} \tag{12}$$

$$\mathbf{f} = \mathbf{V}_f\tilde{\mathbf{f}} \tag{13}$$

Substituting eqns (12) and (13) into the equilibrium equation $\mathbf{Hf} = \mathbf{p}$:

$$\mathbf{H}\mathbf{V}_f\tilde{\mathbf{f}} = \mathbf{V}_p\tilde{\mathbf{p}} \tag{14}$$

Multiplying both sides of eqn (14) by $\mathbf{V}_p^T$:

$$\left(\mathbf{V}_p^T\mathbf{H}\mathbf{V}_f\right)\tilde{\mathbf{f}} = \tilde{\mathbf{p}} \tag{15}$$

The block-diagonalised equilibrium matrix is therefore:

$$\tilde{\mathbf{H}} = \left(\mathbf{V}_p^T\mathbf{H}\mathbf{V}_f\right) \tag{16}$$

and the symmetry-adapted equilibrium equation is now:

$$\tilde{\mathbf{H}}\tilde{\mathbf{f}} = \tilde{\mathbf{p}} \tag{17}$$

In a similar way the block-diagonalised compatibility and flexibility matrices are:

$$\tilde{\mathbf{C}} = \left(\mathbf{V}_f^T\mathbf{C}\mathbf{V}_p\right) \tag{18}$$

$$\tilde{\mathbf{R}} = \left(\mathbf{V}_f^T\mathbf{R}\mathbf{V}_f\right) \tag{19}$$

For the example structure shown in Fig. 1, the (6×6) equilibrium matrix $\mathbf{H}$, written in the original

load and bar-force vector coordinate systems, is given by:

$$\mathbf{H} = \begin{bmatrix} -\sqrt{3}/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & -1 & 0 & 0 & \sqrt{3}/2 & 0 \\ -1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}/2 & 1/2 \\ 0 & 0 & -1 & 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \tag{20}$$

Substituting eqns (11), (20) and (5) into eqn (16), the block-diagonalised equilibrium matrix $\tilde{\mathbf{H}}$ is:

$$\tilde{\mathbf{H}} = \begin{bmatrix} -\sqrt{3} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{10} & \sqrt{3/10} & 0 & 0 \\ 0 & 0 & \sqrt{9/10} & -\sqrt{6/5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\sqrt{10} & \sqrt{3/10} \\ 0 & 0 & 0 & 0 & \sqrt{9/10} & -\sqrt{6/5} \end{bmatrix} \tag{21}$$

Eqn (21) shows that the block-diagonalised equilibrium matrix $\tilde{\mathbf{H}}$ consists of a number of independent submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$:

$$\tilde{\mathbf{H}} = \begin{bmatrix} \tilde{\mathbf{H}}^{(A_1)} & & & \\ & \tilde{\mathbf{H}}^{(A_2)} & & \\ & & \tilde{\mathbf{H}}^{(E)_1} & \\ & & & \tilde{\mathbf{H}}^{(E)_2} \end{bmatrix} \tag{22}$$

Each of the equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$ operates on symmetry-adapted load and bar-force vectors $\tilde{\mathbf{p}}^{(\mu)i}$ and $\tilde{\mathbf{f}}^{(\mu)i}$ in the corresponding symmetry subspaces $\mathbb{V}_{\mathrm{p}}^{(\mu)i}$ and $\mathbb{V}_{\mathrm{f}}^{(\mu)i}$. Each submatrix block $\tilde{\mathbf{H}}^{(\mu)i}$ can be solved separately to give the induced bar-force vectors in equilibrium with the applied load vectors. The original full problem $\mathbf{Hf} = \mathbf{p}$, defined by the original coordinate systems of Figs. 2 and 3, has been decomposed into four independent subproblems which consider the relationship between load and bar-force vectors that have a particular type of symmetry:

$$\tilde{\mathbf{H}}^{(\mu)i}\tilde{\mathbf{f}}^{(\mu)i} = \tilde{\mathbf{p}}^{(\mu)i} \tag{23}$$

Rather than block-diagonalising the entire equilibrium matrix, the equilibrium blocks $\tilde{\mathbf{H}}^{(\mu)i}$ in the independent subproblems can be calculated separately, and are given by:

$$\tilde{\mathbf{H}}^{(\mu)i} = \mathbf{V}_{\mathrm{p}}^{(\mu)i^{\mathrm{T}}}\mathbf{H}\mathbf{V}_{\mathrm{f}}^{(\mu)i} \tag{24}$$

Indeed, it is possible to go further and use sub-structuring techniques to generate the equilibrium blocks $\tilde{\mathbf{H}}^{(\mu)i}$ directly from the equilibrium matrix of a repeating symmetry substructure. A simple sub-structuring technique, based on work done by Healey and Treacy (1991), is described in Kangwai (1997), and Bossavit (1991), develops his method directly from a symmetry sub-structure. Similar subproblems can also be written for the compatibility and flexibility relationships.

Note that, in contrast to a block-diagonalised stiffness matrix, the equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$ are, in general, not square. Indeed, eqn (21) relies on a definition of an empty matrix which may have rows but no columns, or alternatively columns but no rows. In this example, the submatrix block $\tilde{\mathbf{H}}^{(A_2)}$ has no columns, since the symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_2)}$ is empty, but has one row since the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$ is 1-dimensional. This definition fits in well with the further discussion of the equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$ in the remainder of this paper.

As far as we know, the only other description of an empty matrix is in the user manual for the MATLAB program (Math Works, 1996) and the definition given above is compatible to this. Indeed, the user manual states 'MATLAB 5 provides for matrices where one but not all, of the dimensions is zero. The basic model for empty matrices is that any operation that is defined by $m \times n$ matrices, and that produces a result whose dimension is some function of m and n, should still be allowed when m or n is zero'.

5. Implications of block-diagonalisation

Using the symmetry of a structure to block-diagonalise equilibrium, compatibility and flexibility matrices provides useful results. This section will show how block-diagonalising the equilibrium matrix simplifies finding and classifying the states of self-stress and mechanisms in a structure. It will also show that the analysis of a structure using the Force Method can be reduced to the analysis of the independent equilibrium, compatibility and flexibility submatrix blocks, $\tilde{\mathbf{H}}^{(\mu)i}$, $\tilde{\mathbf{C}}^{(\mu)i}$ and $\tilde{\mathbf{R}}^{(\mu)i}$, respectively.

5.1. Analysis of the block-diagonalised equilibrium matrix

In general, an equilibrium matrix $\mathbf{H}$ is an $(m \times n)$ matrix of rank r. From the equilibrium matrix $\mathbf{H}$ it is possible to find the states of self-stress and the inextensional mechanisms of a structure (Pellegrino and Calladine, 1986).

A statically indeterminate structure will have $(m - r)$ state of self-stress. States of self-stress exist when there are bar-force vectors $\mathbf{f}$ in equilibrium with zero load vectors $\mathbf{p}$, i.e. the states of self-stress are all bar-force vectors $\mathbf{f}$ which satisfy the following equation:

$$\mathbf{Hf} = \mathbf{0} \tag{25}$$

Hence, any states of self-stress present in a structure are given by the nullspace of $\mathbf{H}$.

A kinematically indeterminate structure will have $(n - r)$ inextensional mechanisms. Inextensional mechanisms exist when there are displacement vectors $\mathbf{d}$ that are compatible with zero bar elongations, i.e. the inextensional mechanisms are all displacement vectors $\mathbf{d}$ which satisfy the following equation:

$$\mathbf{Cd} = \mathbf{0} \tag{26}$$

Hence, any inextensional mechanisms present in a structure are given by the nullspace of $\mathbf{C}$ (which corresponds to the left-nullspace of $\mathbf{H} = \mathbf{C}^{\mathrm{T}}$).

In Section 4, the load and bar-force vectors $\mathbf{p}$ and $\mathbf{f}$ are transformed into equivalent vectors $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{f}}$

defined by the symmetry-adapted vector bases $\mathbf{V}_\mathrm{p}$ and $\mathbf{V}_\mathrm{f}$ respectively, and hence the equilibrium matrix $\mathbf{H}$ is block-diagonalised into a number of equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$ which are also $(m \times n)$ matrices of rank r. The above analysis for states of self-stress and inextensional mechanisms in the equilibrium matrix $\mathbf{H}$ can be simply carried over to these independent equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$.

The four equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$ defined in eqn (21) for the example structure in Fig. 1, are now examined.

5.1.1. State of self-stress in the first equilibrium submatrix block

$$\tilde{\mathbf{H}}^{(A_1)} = [-1.7321 \quad 1]$$

The (1×2) equilibrium submatrix $\tilde{\mathbf{H}}^{(A_1)}$, corresponding to the first irreducible matrix representation $\Gamma^{(A_1)}$, is of rank 1, and hence will have a state of self-stress present in the corresponding bar-force vector symmetry subspace $\mathbb{V}_\mathrm{f}^{(A_1)}$. The state of self-stress is a bar-force vector $\tilde{\mathbf{f}}_\mathrm{s}^{(A_1)}$, in equilibrium with zero load vectors:

$$\tilde{\mathbf{H}}^{(A_1)}\tilde{\mathbf{f}}_s^{(A_1)} = \mathbf{0} \tag{27}$$

and is given by the nullspace of $\tilde{\mathbf{H}}^{(A_1)}$:

$$\tilde{\mathbf{f}}_\mathrm{s}^{(A_1)} = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \tag{28}$$

This can be transformed back to the original coordinate system:

$$\mathbf{f}_\mathrm{s}^{(A_1)} = \mathbf{V}_\mathrm{f}^{(A_1)}\tilde{\mathbf{f}}_\mathrm{s}^{(A_1)} \tag{29}$$

where $\mathbf{V}_\mathrm{s}^{(A_1)}$ is the bar-force vector basis for the symmetry subspace $\mathbb{V}_\mathrm{f}^{(A_1)}$, defined in eqn (8).

$$\mathbf{f}_\mathrm{s}^{(A_1)} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{12} \\ -1/2 \\ -1/\sqrt{12} \\ -1/2 \\ -1/\sqrt{12} \\ -1/2 \end{bmatrix} \tag{30}$$

This state of self-stress is shown in Fig. 6. The state of self-stress has the full symmetry of the example structure, i.e. three-fold rotation symmetry about the symmetry axis and reflection symmetry in planes a, b and c. This must obviously be the case, as it originates from the symmetry subspace which corresponds to these properties.

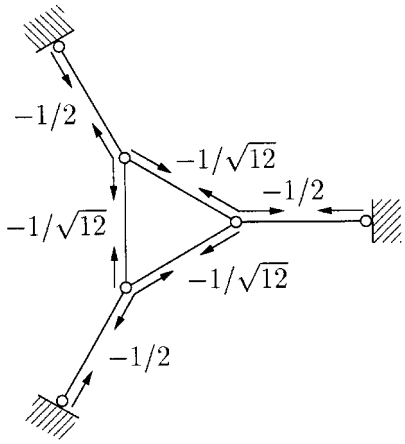

Fig. 6. State of self-stress in the bar-force symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_1)}$.

5.1.2. Mechanism in the second equilibrium submatrix block

The (1×0) equilibrium submatrix $\tilde{\mathbf{H}}^{(A_2)}$, corresponding to the second irreducible matrix representation $\Gamma^{(A_2)}$, is of rank 0, and hence there will be a load vector in the corresponding load vector symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$ which cannot be equilibrated. The mechanism is a displacement vector $\tilde{\mathbf{d}}_{\mathrm{m}}^{(A_2)}$, which is compatible with zero bar elongations. However, since the bar-force vector symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_2)}$ is an empty space, any displacement vector $\tilde{\mathbf{d}}^{(A_2)}$ must be a mechanism. This mechanism is given by the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$ and a vector basis $\mathbf{V}_{\mathrm{p}}^{(A_2)}$ in the original Cartesian coordinate system is given by eqn (5):

$$\mathbf{d}_{\mathrm{m}}^{(A_2)} = \begin{bmatrix} 1/2 \\ 1/\sqrt{12} \\ 0 \\ -1/\sqrt{3} \\ -1/2 \\ 1/\sqrt{12} \end{bmatrix} \tag{31}$$

The mechanism is shown in Fig. 7. The mechanism has only the three-fold rotation symmetry about the symmetry axis of the example structure, and no reflection symmetry.

In general, any $(m \times 0)$ empty equilibrium submatrix $\tilde{\mathbf{H}}^{(\mu)i}$ which operates on an m-dimensional load vector symmetry subspace and a corresponding zero-dimensional bar-force vector symmetry subspace, represents a set of m load vectors which cannot induce any bar-force vectors in the structure, and hence these m load vectors correspond to m independent internal mechanisms.

Similarly, any $(0 \times n)$ empty equilibrium submatrix $\tilde{\mathbf{H}}^{(\mu)i}$ which operates on a zero-dimensional load vector symmetry subspace and a corresponding n-dimensional bar-force vector symmetry subspace, represents a set of n bar-force vectors which are not in equilibrium with any applied load vectors and hence are n independent states of self-stress.

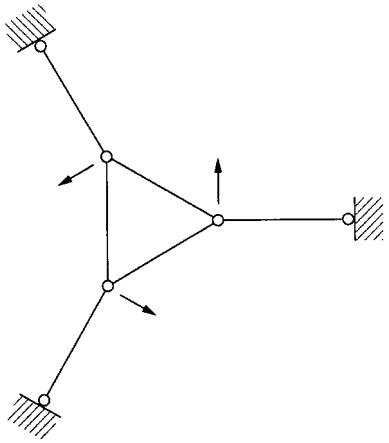

Fig. 7. Mechanism in the displacement symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$.

5.1.3. Third and fourth equilibrium submatrix blocks

The third and fourth (2×2) equilibrium submatrices $\tilde{\mathbf{H}}^{(E)_1}$ and $\tilde{\mathbf{H}}^{(E)_2}$, corresponding to the third irreducible matrix representation $\Gamma^{(E)}$, are both of full rank and hence do not contain any further states of self-stress or inextensional mechanisms.

5.2. Simplification of the Force Method

The Force Method can be used to completely analyse the example structure to give the bar forces, bar elongations and displacements resulting from an applied loading, see, e.g. Livesley (1975). For a block-diagonalised equilibrium matrix $\tilde{\mathbf{H}}$, the analysis can be carried out on the independent equilibrium submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$.

For example, consider the example structure subject to the following loading using the original Cartesian coordinate system shown in Fig. 2:

$$\mathbf{p}^{(A_1)} = \begin{bmatrix} 1/\sqrt{12} \\ -1/2 \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{12} \\ 1/2 \end{bmatrix} \tag{32}$$

which can be transformed into:

$$\tilde{\mathbf{p}}^{(A_1)} = 1 \tag{33}$$

where $\tilde{\mathbf{p}}^{(A_1)}$ is a load vector described by the symmetry-adapted vector basis $\mathbf{V}_{\mathrm{p}}^{(A_1)}$ of the first symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_1)}$. What is the induced bar-force vector $\tilde{\mathbf{f}}^{(A_1)}$?

In Section 5.1.1 it was shown that the first equilibrium submatrix $\tilde{\mathbf{H}}^{(A_1)}$ contains a state of self-stress, and hence the induced bar-force vector $\tilde{\mathbf{f}}^{(A_1)}$ is given by:

$$\tilde{\mathbf{f}}^{(A_1)} = \tilde{\mathbf{f}}_{\mathrm{o}}^{(A_1)} + \tilde{\mathbf{f}}_{\mathrm{s}}^{(A_1)} x \tag{34}$$

where $\tilde{\mathbf{f}}_{\mathrm{o}}^{(A_1)}$ is the bar-force vector in equilibrium with the applied load vector, $\tilde{\mathbf{f}}_{\mathrm{s}}^{(A_1)}$ is the state of self-stress, x is the unknown magnitude of the self-stress.

The state of self-stress $\tilde{\mathbf{f}}_{\mathrm{s}}^{(A_1)}$ was found in Section 5.1.1. The bar-force vector $\tilde{\mathbf{f}}_{\mathrm{o}}^{(A_1)}$ is a particular solution of the equilibrium equation:

$$\tilde{\mathbf{H}}^{(A_1)} \tilde{\mathbf{f}}_{\mathrm{o}}^{(A_1)} = \tilde{\mathbf{p}}^{(A_1)} \tag{35}$$

for which a possible choice is:

$$\tilde{\mathbf{f}}_{\mathrm{o}}^{(A_1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{36}$$

To find the magnitude of the state of self-stress x, the following equation ensures that compatibility is satisfied (Livesley, 1975):

$$\tilde{\mathbf{f}}_{\mathrm{s}}^{(A_1)\mathrm{T}} \tilde{\mathbf{R}}^{(A_1)} \tilde{\mathbf{f}}_{\mathrm{s}}^{(A_1)} x = -\tilde{\mathbf{f}}_{\mathrm{s}}^{(A_1)\mathrm{T}} \tilde{\mathbf{R}}^{(A_1)} \tilde{\mathbf{f}}_{\mathrm{o}}^{(A_1)} \tag{37}$$

where $\tilde{\mathbf{R}}^{(A_1)}$ is the matrix of member flexibilities for the symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_1)}$. The matrix of member flexibilities $\mathbf{R}$ for the original internal vector space $\mathbb{V}_{\mathrm{f}}$ is:

$$\mathbf{R} = \frac{l}{AE} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

where l is the bar length, A is the cross-sectional area of the bar, E is the Young's modulus.

The matrix of member flexibilities for the symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_1)}$ is:

$$\tilde{\mathbf{R}}^{(A_1)} = \mathbf{V}_{\mathrm{f}}^{(A_1)\mathrm{T}} \mathbf{R} \mathbf{V}_{\mathrm{f}}^{(A_1)} \tag{39}$$

$$\tilde{\mathbf{R}}^{(A_1)} = \frac{l}{AE} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{40}$$

Hence, solving eqn (37), the magnitude of the state of self-stress is:

$$x = \sqrt{3}/2 \tag{41}$$

Substituting eqns (36), (28) and (41) into eqn (34), the induced bar-force vector $\tilde{\mathbf{f}}^{(A_1)}$ due to the applied load is:

$$\tilde{\mathbf{f}}^{(A_1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \sqrt{3}/2 = \begin{bmatrix} -\sqrt{3/16} \\ 1/4 \end{bmatrix} \tag{42}$$

The induced bar-force vector $\mathbf{f}^{(A_1)}$ in the original bar-force coordinate system of Fig. 3 is:

$$\mathbf{f}^{(A_1)} = \mathbf{V}_{\mathrm{f}}^{(A_1)} \tilde{\mathbf{f}}^{(A_1)} \tag{43}$$

$$\mathbf{f}^{(A_1)} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -\sqrt{3/16} \\ 1/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/\sqrt{48} \\ -1/4 \\ 1/\sqrt{48} \\ -1/4 \\ 1/\sqrt{48} \end{bmatrix} \tag{44}$$

The induced bar-elongation vector $\tilde{\mathbf{e}}^{(A_1)}$ can now be found using the stress–strain relationship of eqn (3):

$$\tilde{\mathbf{e}}^{(A_1)} = \tilde{\mathbf{R}}^{(A_1)} \tilde{\mathbf{f}}^{(A_1)} \tag{45}$$

$$\tilde{\mathbf{e}}^{(A_1)} = \frac{l}{AE} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} -\sqrt{3/16} \\ 1/4 \end{bmatrix} = \frac{l}{AE} \begin{bmatrix} -3/4 \\ \sqrt{3/16} \end{bmatrix} \tag{46}$$

The induced bar-elongation vector $\mathbf{e}^{(A_1)}$ in the original bar-elongation coordinate system of Fig. 3 is:

$$\mathbf{e}^{(A_1)} = \mathbf{F}_f^{(A_1)} \tilde{\mathbf{e}}^{(A_1)} \tag{47}$$

$$\mathbf{e}^{(A_1)} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} \frac{l}{AE} \begin{bmatrix} -3/4 \\ \sqrt{3/16} \end{bmatrix} = \frac{l}{AE} \begin{bmatrix} -\sqrt{3/16} \\ 1/4 \\ -\sqrt{3/16} \\ 1/4 \\ -\sqrt{3/16} \\ 1/4 \end{bmatrix} \tag{48}$$

The induced displacement vector $\tilde{\mathbf{d}}^{(A_1)}$ is given by the kinematic compatibility relationship of eqn (2):

$$\tilde{\mathbf{H}}^{(A_1)\mathrm{T}} \tilde{\mathbf{d}}^{(A_1)} = \tilde{\mathbf{e}}^{(A_1)} \tag{49}$$

for which the solution is:

$$\tilde{\mathbf{d}}^{(A_1)} = \frac{\sqrt{3}l}{4AE} \tag{50}$$

The induced displacement vector $\mathbf{d}^{(A_1)}$ in the original coordinate system of Fig. 2 is:

$$\mathbf{d}^{(A_1)} = \mathbf{V}_{\mathrm{p}}^{(A_1)} \tilde{\mathbf{d}}^{(A_1)} \tag{51}$$

$$\mathbf{d}^{(A_1)} = \begin{bmatrix} 1/\sqrt{12} \\ -1/2 \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{12} \\ 1/2 \end{bmatrix} \frac{\sqrt{3}l}{4AE} = \frac{l}{AE} \begin{bmatrix} 1/8 \\ -\sqrt{3/64} \\ -1/4 \\ 0 \\ 1/8 \\ \sqrt{3/64} \end{bmatrix} \tag{52}$$

Eqn (64) gives the displacement vector $\mathbf{d}^{(A_1)}$ compatible with the bar-elongation vector $\mathbf{e}^{(A_1)}$, which is induced by the load vector $\mathbf{p}^{(A_1)}$. However a general solution displacement vector $\mathbf{d}$ may include some displacement component $\mathbf{d}_{\mathrm{m}}^{(A_2)}$ of unknown magnitude y from the second symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$, and therefore the general solution displacement vector $\mathbf{d}$ is given by:

$$\mathbf{d} = \mathbf{d}^{(A_1)} + y\mathbf{d}_{\mathrm{m}}^{(A_2)} \tag{53}$$

where $\mathbf{d}^{(A_1)}$ is the displacement vector compatible with the bar-elongation vector, $\mathbf{d}_{\mathrm{m}}^{(A_2)}$ is the inextensional mechanism, y is the magnitude of the inextensional mechanism.

6. Example: states of self-stress and mechanisms of a Geiger dome

The aim of this section is to show that the methods described in this paper can be applied to the analysis of more complex structures. In particular, it looks at the states of self-stress and mechanisms of a Geiger dome.

A Geiger dome is a class of cable-and-strut prestressed tensegrity dome, which have been proposed by D.H. Geiger. Larger domes of this type have been built, an example being the Sun Coast Dome in Florida, which has a diameter of 210 m. There have been a number of studies of the structural behaviour of a Geiger dome under general loading conditions. For example, Pellegrino (1992) has investigated how the single state of self-stress stiffens all the inextensional mechanisms within a simplified version of a Geiger dome with only four-fold rotation and reflection symmetry, shown in Fig. 8. This example uses the simplified Geiger dome to show how the methods described in this paper considerably simplify finding and classifying states of self-stress and mechanisms in a structure. Of course similar calculations could be carried out for more complex Geiger domes, but for clarity only the simplified Geiger dome is analysed here. In the original work by Pellegrino, finding 'simple' mechanisms was not an easy task, whereas using the methods in this paper a sensible symmetry classification is automatically found.

The Geiger dome is transformed into an equivalent structure by the following set of symmetry operations: the identity, symmetry operation E; rotation by 90, 180 or 270° about the vertical axis through the origin O, symmetry operations C_4, C_4^2 or C_4^3; reflection in the vertical planes x, y, m or n, symmetry operations σ_x, σ_y, σ_m or σ_n. These eight symmetry operations constitute the symmetry group C_{4v}. The irreducible matrix representations $\Gamma^{(\mu)}$ for this symmetry group are shown in Table 3.

Using the method described in Section 3, both the load and bar-force vector spaces, $\mathbb{V}_{\mathrm{p}}$ and $\mathbb{V}_{\mathrm{f}}$, can each be decomposed into six symmetry subspaces, since there are four 1-D and one 2-D irreducible matrix representations for the symmetry group $\mathbf{C}_{4v}$. Using these symmetry-adapted coordinate systems,

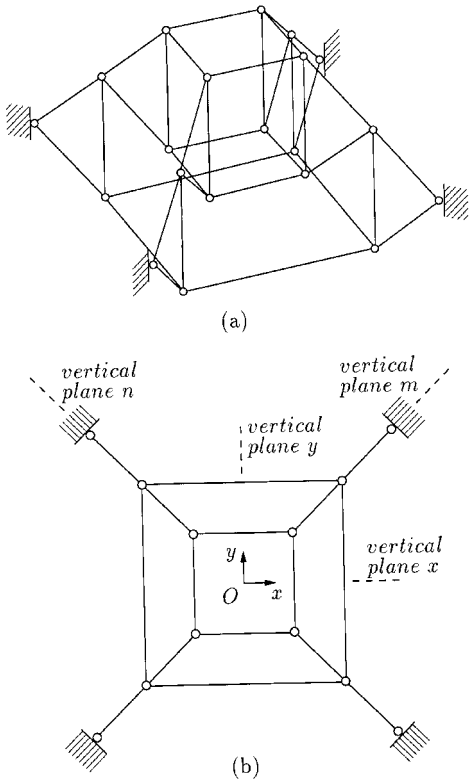

Fig. 8. Simplified Geiger dome, with $\mathbf{C}_{4v}$ symmetry. (a) 3-D view; (b) plan view, showing planes of symmetry.

the block-diagonalised equilibrium matrix has the following form:

$$\tilde{\mathbf{H}} = \begin{bmatrix} \tilde{\mathbf{H}}^{(A_1)} \ (8\times 9) & & & & \\ & \tilde{\mathbf{H}}^{(A_2)}\,(4\times 0) & & & \\ & & \tilde{\mathbf{H}}^{(B_1)}\,(4\times 3) & & \\ & & & \tilde{\mathbf{H}}^{(B_2)}\,(8\times 6) & \\ & & & & \tilde{\mathbf{H}}^{(E)_1} \ (12\times 9) \\ & & & & & \tilde{\mathbf{H}}^{(E)_2} \ (12\times 9) \end{bmatrix} \tag{54}$$

Table 3
Irreducible representations of symmetry group $\mathbf{C}_{4v}$

$\mathbf{C}_{4v}$	E	C_4	C_4^2	C_4^3	σ_x	σ_y	σ_m	σ_n
$\Gamma^{(A_1)}$	1	1	1	1	1	1	1	1
$\Gamma^{(A_2)}$	1	1	1	1	−1	−1	−1	−1
$\Gamma^{(B_1)}$	1	−1	1	−1	1	1	−1	−1
$\Gamma^{(B_2)}$	1	−1	1	−1	−1	−1	1	1
$\Gamma^{(E)}$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

Due to the size of the matrices, $\tilde{\mathbf{H}}$, $\mathbf{V}_p$ and $\mathbf{V}_f$ are not shown in full. However, the next two sections will examine the states of self-stress and mechanisms found in the different submatrix blocks $\tilde{\mathbf{H}}^{(\mu)i}$.

6.1. States of self-stress in the Geiger dome

The first equilibrium submatrix block $\tilde{\mathbf{H}}^{(A_1)}$ is an (8×9) matrix of rank 8. Hence, a single state of self-stress exists in the bar-force symmetry subspace $\mathbb{V}_f^{(A_1)}$ and is shown in Fig. 9.

The bar-force vector symmetry subspace $\mathbb{V}_f^{(A_1)}$ has the fully symmetry of the Geiger dome, i.e. bar-force vectors in this symmetry subspace are left unchanged by all the operations of the symmetry group $\mathbf{C}_{4v}$.

The remaining five bar-force vector symmetry subspace of the Geiger dome do not contain any states of self-stress.

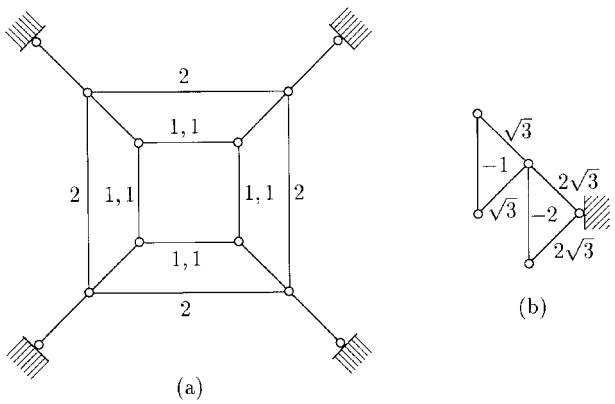

Fig. 9. Bar-forces of the state of self-stress in the bar-force symmetry subspace $\mathbb{V}_f^{(A_1)}$. (a) Plan view, [1, 1] indicates there is a unit magnitude tensile force in both the upper and lower inner rings. (b) Elevation of one radial truss section—all four of these have identical bar-forces.

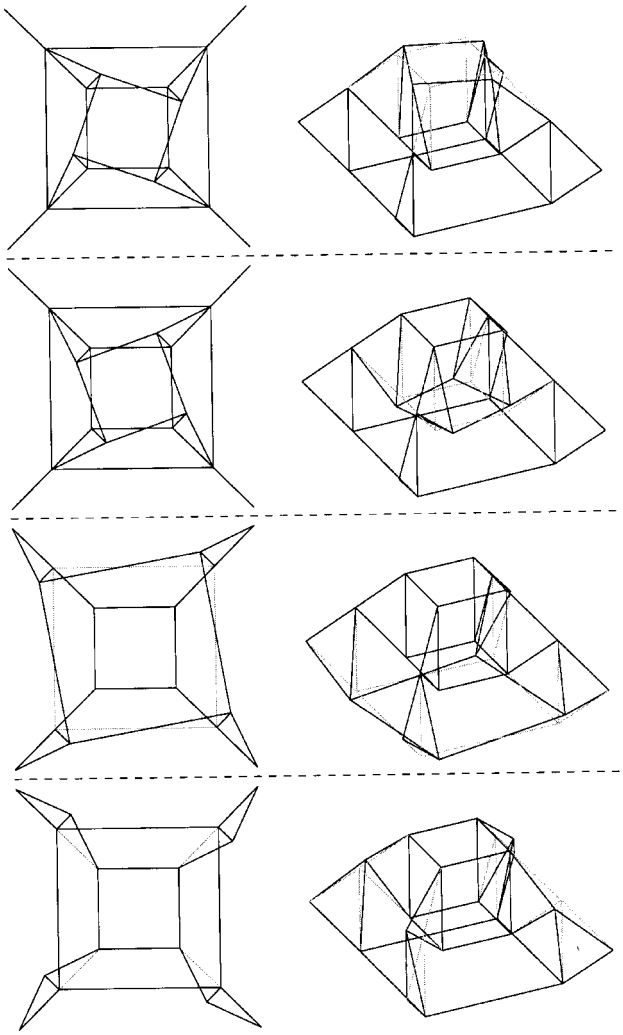

Fig. 10. Four mechanisms in the displacement symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$, with $\mathbf{C}_4$ symmetry properties.

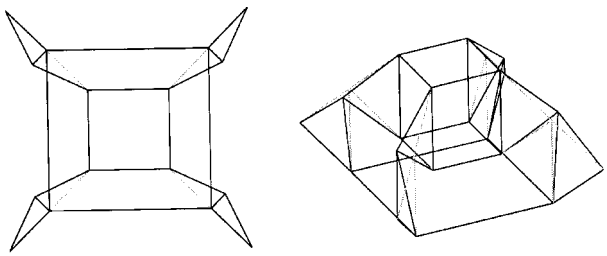

Fig. 11. Mechanism in the displacement symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(B_1)}$, with $\mathbf{C}_{2v}$ symmetry properties (reflection in planes x and y).

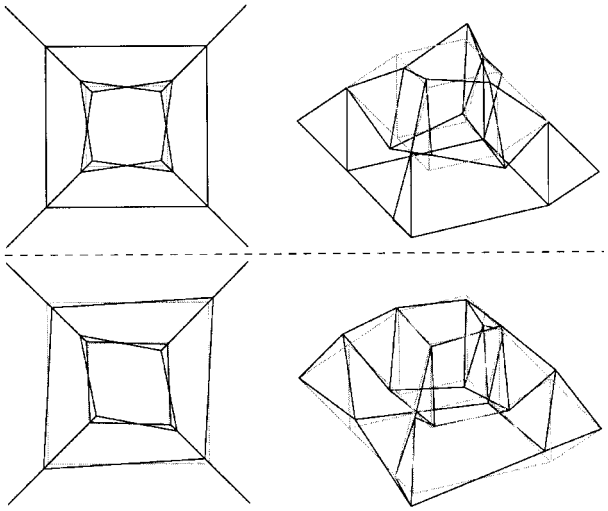

Fig. 12. Two mechanisms in the displacement symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(B_2)}$, with $\mathbf{C}_{2v}$ symmetry properties (reflection in planes m and n).

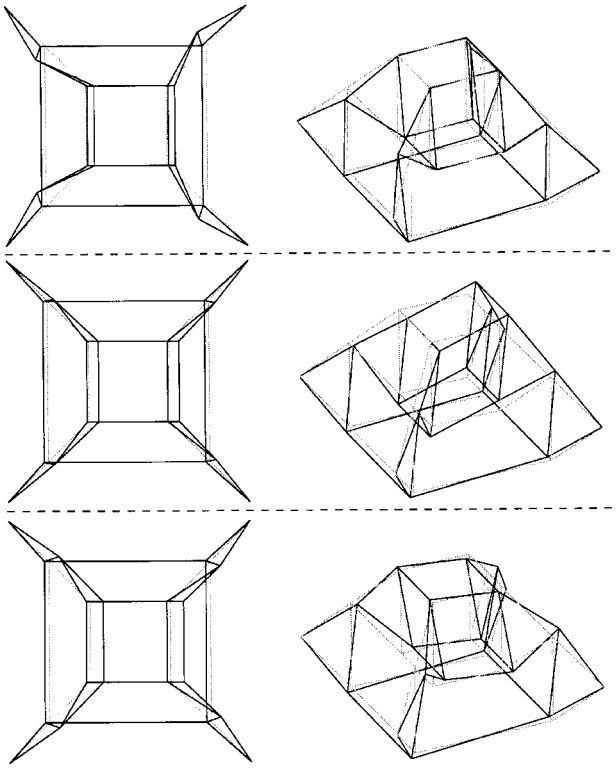

Fig. 13. Three mechanisms in the displacement symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(E)_1}$, with $\mathbf{C}_s$ symmetry properties (reflection in plane x).

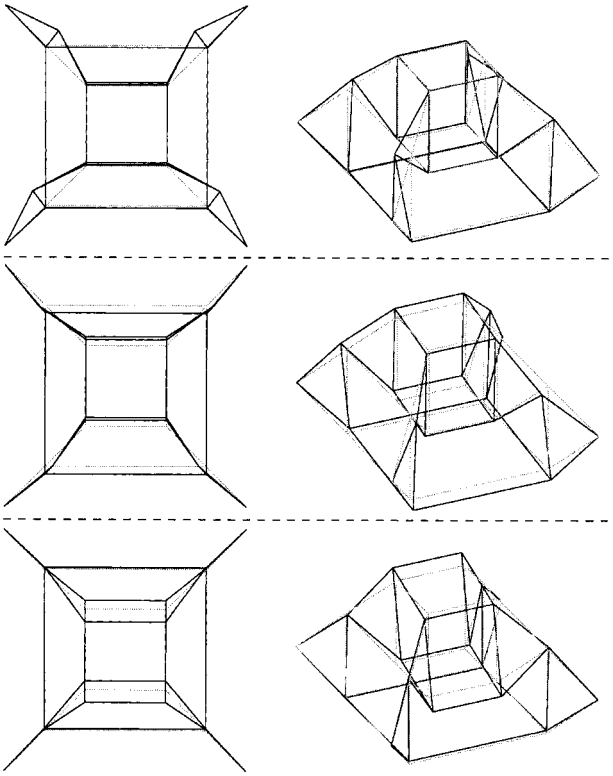

Fig. 14. Three mechanisms in the displacement symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(E)_2}$, with $\mathbf{C}_s$ symmetry properties (reflection in plane y).

6.2. Mechanisms in the Geiger dome

The full equilibrium matrix $\mathbf{H}$ is a (48×36) matrix of rank 35, hence there exist 13 inextensional mechanisms in the Geiger dome (Pellegrino, 1992).

The second equilibrium submatrix block $\tilde{\mathbf{H}}^{(A_2)}$, which operates on the symmetry subspaces $\mathbb{V}_{\mathrm{p}}^{(A_2)}$ and $\mathbb{V}_{\mathrm{f}}^{(A_2)}$, is a (4×0) matrix of rank 0. The bar-force symmetry subspace $\mathbb{V}_{\mathrm{f}}^{(A_2)}$ is, therefore, an empty subspace and hence the four displacement vectors in the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(A_2)}$ must be inextensional mechanisms. These four mechanisms are given by the four basis vectors in $\mathbf{V}_{\mathrm{p}}^{(A_2)}$ and are shown in Fig. 10. We can see that each of these mechanisms has the full rotation symmetry of the Geiger dome but none of the reflection symmetry, i.e. $\mathbf{C}_4$ symmetry properties.

The third equilibrium submatrix block $\tilde{\mathbf{H}}^{(B_1)}$ is a (4×3) matrix of rank 3. Hence, an inextensional mechanism exists in the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(B_1)}$, and is shown in Fig. 11. This mechanism has only $\mathbf{C}_{2v}$ symmetry properties, with reflection symmetry in the x- and y-planes.

The fourth equilibrium submatrix block $\tilde{\mathbf{H}}^{(B_2)}$ is an (8×6) matrix of rank 6. Hence, two inextensional mechanisms exist in the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(B_2)}$, and are shown in Fig. 12. They also have only $\mathbf{C}_{2v}$ symmetry properties, but now with reflection symmetry in the m and n planes.

The fifth equilibrium submatrix block $\tilde{\mathbf{H}}^{(E)_1}$ is a (12×9) matrix of rank 9. Hence, three inextensional mechanisms exist in the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(E)_1}$ and are shown in Fig. 13. These mechanisms have only reflection symmetry in the x-plane, i.e. $\mathbf{S}_2$ symmetry properties.

The sixth equilibrium submatrix block $\tilde{\mathbf{H}}^{(E)_2}$ is a (12×9) matrix of rank 9. Hence, three inextensional

mechanisms exist in the symmetry subspace $\mathbb{V}_{\mathrm{p}}^{(E)_2}$ and are shown in Fig. 14. These mechanisms have only reflection symmetry in the y-plane, i.e. $\mathbf{S}_2$ symmetry properties.

7. Conclusions

By defining both external and internal symmetry-adapted coordinate systems, we have shown that the equilibrium matrix of a symmetric structure can be block-diagonalised, and that this can give useful insight into the structural response of such a system, as well as simplifying computations.

References

Altmann, S.L., Herzig, P., 1994. Point-Group Theory Tables. Clarendon Press, Oxford.

Bossavit, A., 1993. On the computation of strains and stresses in symmetrical articulated structures. Lectures in Applied Mathematics, American Mathematical Society 29, 111–123.

Calladine, C.R., 1978. Buckminster Fuller's 'Tensegrity' structures and Clerk Maxwell's rules for the construction of stiff frames. Int. J. Solids Structures 14, 161–172.

Dinkevich, S., 1991. Finite symmetric systems and their analysis. Int. J. Solids Structures 27 (10), 1215–1253.

Healey, T.J., Treacy, J.A., 1991. Exact block diagonalization of large eigenvalue problems for structures with symmetry. Int. J. Numerical Methods in Engineering 31, 265–285.

Kangwai, R.D., 1997. The Analysis of Symmetric Structures using Group Representation Theory. Ph.D. thesis, University of Cambridge.

Kangwai, R.D., Guest, S.D., Pellegrino, S., 1999. An introduction to the analysis of symmetric structures. Computers and Structures 71, 671–688.

Livesley, R.K., 1975. Matrix Methods of Structural Analysis. Pergamon Press.

McGuire, W., Gallagher, R.H., 1979. Matrix Structural Analysis. John Wiley & Sons.

MathWorks, 1996. The Language of Technical Computing, Using MATLAB, Version 5. The MathWorks.

Pellegrino, S., 1992. A class of tensegrity domes. Int. J. Space Structures 7, 127–142.

Pellegrino, S., 1993. Structural computations with the singular-value decomposition of the equilibrium matrix. Int. J. Solids Structures 30, 3025–3035.

Pellegrino, S., Calladine, C.R., 1986. Matrix analysis of statically and kinematically indeterminate frameworks. Int. J. Solids Structures 22, 409–428.

Strang, G., 1986. Introduction to Applied Mathematics. Wellesley-Cambridge Press.

Wilson, E.B., Decius, J.C., Cross, P.C., 1955. Molecular Vibrations. Dover, N.Y.

Zhong, W., Qiu, C., 1983. Analysis of symmetric or partially symmetric structures. Computer Methods in Applied Mechanics and Engineering 38, 1–18.

Zloković, G., 1989. Group Theory and G-Vector Spaces in Structural Analysis, Vibration, Stability and Statics. Ellis Horwood.

Zloković, G., 1992. Group Supermatrices in Finite Element Analysis. Ellis Horwood.